GROUP CLASSIFICATION AND INVARIANT SOLUTIONS FOR THE EQUATIONS OF FLOW AND HEAT TRANSFER OF A VISCOPLASTIC MEDIUM

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Exact solutions without major restrictions on the properties of the material are needed in research on the flow (especially viscosity) of metals at high temperatures under nonisothermal conditions. Often the shear resistance is governed mainly by the temperature and the deformation rate. Here are examined the group properties of the equations of flow and heat transfer of a medium whose shear resistance is a function of temperature and rate of shear deformation. The properties specific to metals are not used, so the results are applicable to a variety of media.

§1. Here we consider three types of flow accompanied by heat transfer for a medium filling a finite or infinite region $x > x_0$ ($x_0 \geq 0$).

1. Planar rectangular flow without a pressure gradient caused by motion of a boundary in a direction perpendicular to the x axis.

2. Rectilinear flow with axial symmetry without a pressure gradient caused by translational motion of a circular cylinder of radius x_0 in the direction of the generator.

3. Flow caused by the rotation of a circular cylinder of radius x_0 about its axis.

It is assumed that the shear stress is a function of the temperature and deformation rate. These simple types of flow allow one to obtain exact solutions without further assumptions about the properties of the medium.

The equation of motion and the equation of heat flow are

$$\rho\frac{\partial v}{\partial t}-\frac{\partial\Phi(\varepsilon,T)}{\partial\varepsilon}\frac{\partial^2 v}{\partial x^2}+\frac{\partial\Phi(\varepsilon,T)}{\partial T}\frac{\partial T}{\partial x}+\frac{\delta_2}{x}\frac{\partial\Phi(\varepsilon,T)}{\partial\varepsilon}\left(\frac{\partial v}{\partial x}-\frac{v}{x}\right)+\frac{\delta_1+\delta_2}{x}\Phi(\varepsilon,T)=0 \qquad (1.1)$$

$$\rho c\frac{\partial T}{\partial t}-\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)-\frac{\delta_1}{x}\lambda(T)\frac{\partial T}{\partial x}-\rho\frac{\partial L(T)}{\partial t}=0.$$

The above three types of flow correspond to the following combinations of δ_1 and δ_2: 1) $\delta_1 = \delta_2 = 0$; 2) $\delta_1 = 1$, $\delta_2 = 0$; 3) $\delta_1 = \delta_2 = 1$; v is the corresponding velocity component, T is temperature, ρ is density, and c is specific heat. The function $\Phi(\varepsilon, T)$ gives the shear stress τ as a function of T and the deformation rate in shear:

$$\tau=F(\varepsilon,T)\left(\frac{\partial v}{\partial x}-\frac{\delta_2}{x}v\right),\quad \varepsilon=\left|\frac{\partial v}{\partial x}-\frac{\delta_2}{x}v\right|,\quad \Phi=F\varepsilon;$$

this, the thermal conductivity $\lambda(T)$, and the function L(T) (representing heat released by phase transformations) allow a certain range of choice in their forms. It is assumed that the heat produced as a result of the viscosity may be neglected.

We define T°, $f(T^\circ)$, $\Phi^\circ(\varepsilon, T^\circ)$ as follows:

$$T^\circ=\rho\,[cT-L(T)],$$

$$f(T^\circ)=\lambda/\rho\,(c-L'),\qquad \Phi^\circ=\Phi/\rho. \qquad (1.2)$$

System (1.1) becomes as follows in the new variables (the superscript is omitted):

$$(S)\begin{cases}\dfrac{\partial v}{\partial t}+\dfrac{\partial\Phi}{\partial\varepsilon}\dfrac{\partial\varepsilon}{\partial x}+\dfrac{\partial\Phi}{\partial T}\dfrac{\partial T}{\partial x}+\dfrac{\delta_1+\delta_2}{x}\Phi=0,\\[2mm] \dfrac{\partial T}{\partial t}-\dfrac{\partial}{\partial x}\left(f(T)\dfrac{\partial T}{\partial x}\right)-\dfrac{\delta_1}{x}f(T)\dfrac{\partial T}{\partial x}=0,\\[2mm] \dfrac{\partial v}{\partial x}-\dfrac{\delta_2 v}{x}+\varepsilon=0. \end{cases} \qquad (1.3)$$

This is the system that will be examined.

§2. Consider the group properties of system S in accordance with the general methods of [1–3], which have [4–6] been applied to various physical problems. We consider the invariance of S relative to the operator

$$X=\xi^\circ\frac{\partial}{\partial t}+\xi^1\frac{\partial}{\partial x}+\eta^\circ\frac{\partial}{\partial T}+\eta^1\frac{\partial}{\partial v}+\eta^2\frac{\partial}{\partial\theta}+\eta^3\frac{\partial}{\partial\varepsilon}$$

$$\left(\theta=f\frac{\partial T}{\partial x}\right)$$

of the sought group G, the conditions for this being complied with on a manifold specified by S; this gives us a system of equations of the Lie algebra of the basic group, which we write out as follows after simplification:

$$\eta^\circ\frac{1}{f}\frac{df}{dT}-2\frac{\partial\xi'}{\partial x}+\frac{\partial\xi^\circ}{\partial t}=0,\qquad \frac{\partial^2\eta^\circ}{\partial T^2}=0,$$

$$f\frac{\partial\eta^\circ}{\partial x}+\left(\frac{\partial\eta^\circ}{\partial T}+\frac{\partial\xi^1}{\partial x}-\frac{\partial\xi^\circ}{\partial t}\right)\theta-\eta^2=0,$$

$$\frac{\partial\eta^1}{\partial x}-\left(\frac{\partial\eta^1}{\partial v}-\frac{\partial\xi^1}{\partial x}\right)\varepsilon-$$

$$-\frac{\delta_2}{x}\left[\left(\frac{\partial\xi^1}{\partial x}-\frac{\xi'}{x}-\frac{\partial\eta^1}{\partial v}\right)v+\eta^1\right]+\eta^3=0,$$

$$2\frac{\partial^2\eta^\circ}{\partial T\,\partial x}+3\frac{\partial^2\xi'}{\partial x^2}+\frac{1}{f}\frac{\partial\xi^1}{\partial t}+\frac{\delta_1}{x}\left(\frac{\partial\xi'}{\partial x}-\frac{\xi^1}{x}\right)=0,$$

$$\frac{\partial\eta^\circ}{\partial t}-f\frac{\partial^2\eta^\circ}{\partial x^2}-\frac{\delta_1}{x}f\frac{\partial\eta^\circ}{\partial x}=0,$$

$$F_2\frac{\partial\eta^3}{\partial x}-F_1\frac{\partial\eta^\circ}{\partial x}+\frac{\partial\eta^1}{\partial t}-$$

$$-\left(\frac{\delta_2 v}{x}-\varepsilon\right)\frac{\partial\xi^1}{\partial t}+F_2\left(\frac{\delta_2 v}{x}-\varepsilon\right)\frac{\partial\eta^3}{\partial v}+$$

$$+\frac{\delta_1+\delta_2}{x}\left[F_2\eta^3-F_1\eta^\circ-\Phi\left(\frac{\partial\eta^2}{\partial v}+\frac{\xi^1}{x}-\frac{\partial\xi^\circ}{\partial t}\right)\right]=0, \qquad (2.1)$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\Theta - (\ln T)'\frac{\partial\psi}{\partial x} =$$
$$= \frac{1}{RP}\left[\varkappa\Delta\Theta + 2(1+\beta)\varkappa(\ln T)'\frac{\partial\Theta}{\partial y}\right] +$$
$$+ \frac{N}{P}\left[(\gamma - \beta - 1)\frac{1}{R}\frac{\eta U'^2}{T} -\right.$$
$$\left. -(\alpha + \beta + 1)\frac{A}{R_m}\frac{B_x'^2}{\sigma T}\right]\Theta +$$
$$+ 2\frac{N}{PR}\frac{\eta U'^2}{T}\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) + 2\frac{NA}{PR_m}\frac{B_x'}{\sigma T}\Delta\varphi$$
$$\left(R = \frac{\rho l v^*}{\eta_0},\ A = \frac{B_0^2}{\mu\rho v^{*2}},\ R_m = \mu\sigma_0 l v^*,\ P = \frac{\eta_0 c_p}{\varkappa_0}\right). \qquad (2.4)$$

Here R is the Reynolds number, A is the Alfven number, R_m is the magnetic Reynolds number, P is

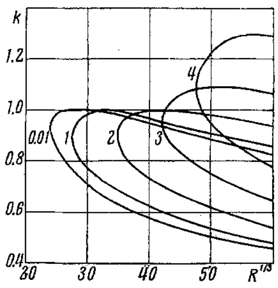


Fig. 1

the Prandtl number, U is the velocity of the unperturbed flow, B is the unperturbed magnetic field, T is the unperturbed temperature, while σ, $\varkappa$, η are the electrical and thermal conductivity and viscosity in the unperturbed flow. The primes denote differentiation with respect to y.

As usual the solution of the system is written in the form

$$\psi = \psi(y)\exp ik(x - ct), \qquad (2.5)$$

where k is the dimensionless wave number and kc is the dimensionless frequency of the oscillations. Equations (2.2)–(2.4) must be solved for the following obvious conditions:

$$\psi(\pm 1) = \psi'(\pm 1) = 0, \qquad \Theta(\pm 1) = 0. \qquad (2.6)$$

The boundary conditions for the magnetic field in the case of nonconducting walls have the form

$$(\varphi'/\varphi)_{\pm 1} = \mp k. \qquad (2.7)$$

If system (2.2)–(2.4) is not separable, then hydrodynamic, electrodynamic, and thermal effects exert a simultaneous influence on the stability.

3. The Overheat Instability. We shall first of all consider the case $S \ll R_m$, where $S = M^2/R$ is the hydromagnetic interaction parameter. Clearly in this case field perturbations caused by the motion of the medium may predominate over velocity perturbations caused by the field. In the limit for $A \to 0$ for $\gamma = 0$ we may imagine a situation when the velocity perturbations also tend to zero, and the terms containing ψ in Eqs. (2.3), (2.4) may be neglected. If we make the further assumption that $R_m \ll 1$, then we have from (2.3)

$$\Delta\varphi = \alpha B_x'\Theta. \qquad (3.1)$$

Using (2.4), (2.5), and (3.1) and neglecting for simplicity the contribution of viscous dissipation and the fact that $\varkappa$ is not constant, we obtain, after making formal transformations,

$$\Theta'' + (E - V)\Theta = 0,$$
$$E = -k^2 + ikcRP,$$
$$V = -\alpha\Pi\frac{j^2}{\sigma T} + ikURP \qquad \left(\Pi = \frac{j^{*2}l^2}{\sigma_0\varkappa_0 T_0}\right). \qquad (3.2)$$

The problem thus becomes one of finding the eigenvalues of the Schrödinger equation with a complex potential V. If the initial steady state is symmetric with respect to y, then it is not hard to see that ReV is a "potential well," and ImV has the form of a hump. The potential may be expanded in a series to give the Schrödinger equation for a harmonic oscillator in the region of the axis of the channel. Having thus ascertained that finite solutions exist [11], we may employ simple approximate methods in order to investigate (3.2). For example in the quasi-classical approximation we replace d/dy by ik_y and obtain the stability criterion immediately (in dimensional form):

$$\varkappa_0 k^2 > \frac{d\ln\sigma}{d\ln T}\frac{j^2}{\sigma T}. \qquad (3.3)$$

Formula (3.3) was obtained previously for the general case in paper [7], but the question of the existence of finite solutions was not considered. The presence of the factor α in inequality (3.3) prompts us to call the instability an overheat instability [5, 7]. For simplicity we shall restrict ourselves to considering the case $S \ll R_m \ll 1$ in the quasi-classical approximation. A similar analysis may be carried out without this last restriction.

4. Hydrodynamic Instability. We shall now consider the other limiting case in which the instability is caused by the purely hydrodynamic mechanism of the untwisting of the velocity gradient vortex. It is well known that the onset of hydrodynamic instability occurs for fairly large Reynolds numbers R. We may therefore neglect the small terms in the right-hand side of (2.2), retaining, however, the old derivative. Further we shall confine ourselves to the case $R_m \ll 1$, where we can neglect terms containing B_x compared with B_0. From Eq. (2.3) we have

$$\varphi'' - k^2\varphi = -R_m\sigma\psi' + \alpha B_x'\Theta. \qquad (4.1)$$

If the hydromagnetic interaction parameter $S \ll \ll 1$, i.e., the Hartmann number is not very large, then we may eliminate φ from (2.2) using (4.1) and, neglecting small terms, finally arrive at a problem which is one of finding the eigenvalues for an Orr–Sommerfeld type equation

$$(U - c)(\psi'' - k^2\psi) - U''\psi = \frac{1}{ikR}\eta\psi^{IV} \qquad (4.2)$$

with boundary conditions (2.6). Thus for $R_m \ll 1$, $S \ll \ll 1$, $\alpha S < 1$ the magnetic field and nonisothermal nature of the flow exert an indirect influence on the stability of the motion, altering the velocity profile and introducing a viscosity profile into Eq. (4.2). In order to solve the problem we use the familiar Heisenberg–Lin method [9]. We shall, as usual, confine ourselves to treating even perturbations over the channel half-

Table of Invariant Solutions

$$n=\frac{m+\alpha}{2m},\qquad \sigma=\frac{\alpha+2\beta}{\alpha},\qquad l=\frac{k+m+\alpha}{2m}$$

	$J_1=\xi$	T	v	$\alpha,\ \beta$	δ
$S,\quad \Phi=\Phi(\varepsilon, T),\quad f=f(T)$					
H_1	x	J_2	J_3	—	
H_2	t	J_2	J_3	—	$\delta_1=0$
H_3	$xt^{-1/2}$	J_2	$t^{1/2}J_3$	—	
H_4	x	J_2	$x^{\delta_2}(J_3+t)$	—	
H_5	t	J_2	$k^{-1}x+J_3$	—	$\delta_1=0$
H_6	$x-t$	J_2	J_3+mt	—	$\delta_1=0$
H_7	$xt^{-1/2}$	J_2	$x(J_3+k\ln x)$	—	$\delta_2=1$
$S_1,\quad \Phi=T^{\alpha+\beta}\Psi(\varepsilon T^{-\beta}),\quad f=T^{\alpha}$					
$H^1_{8,1}$	t	$x^{2/\alpha}J_2$	$x^{\sigma}J_3$	$\alpha\neq 0$	
$H^1_{8,2}$	xt^{-n}	$t^{1/m}J_2$	$t^{n+\beta/m}J_3$	—	
H^1_{9}	$xe^{-\alpha t}$	$e^{2t}J_2$	$e^{\sigma\alpha t}J_3$	—	
H^1_{10}	$te^{2\alpha x}$	$e^{2x}J_2$	$e^{2\beta x}J_3$	—	$\delta_1=0$
H^1_{11}	$x-kt$	$e^{2t}J_2$	$e^{2\beta t}J_3$	$\alpha=0$	$\delta_1=0$
H^1_{12}	$te^{2\alpha x}$	$e^{2x}J_2$	J_3+kx	$\beta=0$	$\delta_1=0$
$H^1_{13,1}$	xt^{-n}	$t^{1/m}J_2$	$x(J_3+{}^1/_2km^{-1}\ln t)$	$\beta=0$ $m\neq 0$	$\delta_2=1$
$H^1_{13,2}$	t	$x^{2/\alpha}J_2$	$x(J_3+k\alpha^{-1}\ln x)$	$\beta=0$ $m=0$	$\delta_2=1$
$H^1_{14,1}$	$xt^{-\beta/\sigma\alpha}$	$t^{-1/\sigma\alpha}J_2$	$J_3-(2\sigma\alpha)^{-1}\ln t$	$\alpha\neq-2\beta$	$\delta_2=0$
$H^1_{14,2}$	t	$x^{-1/\beta}J_2$	$J_3-{}^1/_2\beta^{-1}\ln x$	$\alpha=$ $=-2\beta\neq 0$	$\delta_2=0$
$H^1_{15,1}$	$xe^{-\alpha t}$	$e^{2t}J_2$	J_3+kt	$\alpha=-2\beta$	$\delta_2=0$
$H^1_{15,2}$	$xe^{-\alpha t}$	$e^{2t}J_2$	$x(J_3+kt)$	$\beta=0$	$\delta_2=1$
H^1_{16}	$x-mt$	$e^{2t}J_2$	J_3+kt	$\alpha=\beta=0$	$\delta_1=0$
$S_2,\quad \Phi=e^{(\alpha+\beta)T}\Psi(\varepsilon e^{-\beta T}),\quad f=e^{\alpha T}$					
$H^2_{8,1}$	t	$2\alpha^{-1}\ln x+J_2$	$x^{\sigma}J_3$	$\alpha\neq 0$	
$H^2_{8,2}$	xt^{-n}	$m^{-1}\ln t+J_2$	$t^{n+\beta/m}J_3$	$m\neq 0$	
H^2_{9}	$xe^{-\alpha t}$	$2t+J_2$	$e^{\sigma\alpha t}J_3$	—	
H^2_{10}	$te^{2\alpha x}$	$2x+J_2$	$e^{2\beta x}J_3$	—	$\delta_1=0$
H^2_{11}	$x-kt$	$2t+J_2$	$e^{2\beta t}J_3$	$\alpha=0$	$\delta_1=0$
H^2_{12}	$te^{2\alpha x}$	$2x+J_2$	$kx+J_3$	$\beta=0$	$\delta_1=0$
$H^2_{13,1}$	xt^{-l}	$m^{-1}\ln t+J_2$	$t^{n}J_3$	$\beta=0$ $m\neq 0$	$\delta_2=1$
$H^2_{13,2}$	t	$2(k+\alpha)^{-1}\ln x+$ $+J_2$	$x^{\alpha/(k+\alpha)}J_3$	$\beta=0$ $k\neq-\alpha$	$\delta_2=1$
$H^2_{14,1}$	$x^{-\beta/\sigma\alpha}$	$J_2-(\sigma\alpha)^{-1}\ln t$	$J_3-(2\sigma\alpha)^{-1}\ln t$	$\alpha\neq-2\beta$	$\delta_2=0$
$H^2_{14,2}$	t	$J_2-\beta^{-1}\ln x$	$J_3-(2\beta)^{-1}\ln x$	$\alpha=$ $=-2\beta\neq 0$	$\delta_2=0$
$H^2_{15,1}$	$xe^{-\alpha t}$	$2t+J_2$	J_3+kt	$\alpha=-2\beta$	$\delta_2=0$
$H^2_{15,2}$	$xe^{-\alpha t}$	J_2+2t	$x(J_3+kt)$	$\beta=0$	$\delta_2=1$
H^2_{16}	$x-mt$	J_2+2t	J_3+kt	$\alpha=\beta=0$	$\delta_1=0$

Table of Invariant Solutions (cont'd)

	$J_1=\xi$	T	v	$\alpha,\ \beta$	δ
$S_3,\ \Phi=T^{\alpha}\Psi(\varepsilon-\beta\ln T),\ f=T^{\alpha}$					
$H^3_{8,1}$	t	$x^{2/\alpha}J_2$	$x(J_3-2\beta\alpha^{-1}\ln x)$ $x(J_3-\beta\alpha^{-1}(\ln x)^2)$	$\alpha\neq 0$	$\delta_2=0$ $\delta_2=1$
$H^3_{8,2}$	xt^{-n}	$t^{1/m}J_2$	$x(J_3-\beta m^{-1}\ln t)$ $x(J_3-\beta(mn)^{-1}(\ln x)^2)$	$m\neq 0$ $m\neq-\alpha$ $m\neq 0$	$\delta_2=0$ $\delta_2=1$
$H^3_{8,3}$	x	$t^{-1/\alpha}J_2$	$J_3+\beta\alpha^{-1}x\ln x\ln t$ $x(J_3-2\beta t)$	$m=-\alpha\neq 0$ —	$\delta_2=1$ $\delta_2=0$
$H^3_{9,1}$	$xe^{-\alpha t}$	$e^{2t}J_2$	$x(J_3-\beta\alpha^{-1}(\ln x)^2)$	$\alpha\neq 0$	$\delta_2=1$
$H^3_{9,2}$	x	$e^{2t}J_2$	$J_3-2\beta xt\ln x$	$\alpha=0$	$\delta_2=1$
H^3_{10}	$te^{2\alpha x}$	$e^{2x}J_2$	$J_3-\beta x^2$	—	$\delta_1=0$
H^3_{11}	$x-kt$	$e^{2t}J_2$	$J_3-\beta k^{-1}x^2$	$\alpha=0$	$\delta_1=0$
H^3_{12}	$te^{2\alpha x}$	$e^{2x}J_2$	J_3+kx	$\beta=0$	$\delta_1=0$
$H^3_{13,1}$	t	$x^{2/\alpha}J_2$	$x(J_3+k\alpha^{-1}\ln x)$	$\beta=0$ $\alpha\neq 0$	$\delta_2=1$
$H^3_{13,2}$	xt^{-n}	$t^{1/m}J_2$	$x(J_3+{}^1/_2km^{-1}\ln t)$	$\beta=0$ $m\neq 0$	$\delta_2=1$
$H^3_{14,1}$	x	$t^{-1/\alpha}J_2$	$J_3-{}^1/_2\alpha^{-1}\ln t$	$\beta=0$ $\alpha\neq 0$	$\delta_2=0$
$H^3_{14,2}$	x	$t^{-1/2\alpha}J_2$	$J_3-{}^1/_2\alpha^{-1}x\ln t$	$\beta=0$ $\alpha\neq 0$	$\delta_2=1$
H^3_{15}	$xe^{-\alpha t}$	$e^{2t}J_2$	$x(J_3+kt)$	$\beta=0$	$\delta_2=1$
H^3_{16}	$x-mt$	$e^{2t}J_2$	$kt+J_3$	$\alpha=\beta=0$	$\delta_1=0$
$S_4,\ \Phi=e^{\alpha T}\ \Psi(\varepsilon-\beta T),\ f=e^{\alpha T}$					
$H^4_{8,1}$	t	$2\alpha^{-1}\ln x+J_2$	$x(J_3-2\beta\alpha^{-1}\ln x)$ $x(J_3-\beta\alpha^{-1}(\ln x)^2)$	$\alpha\neq 0$	$\delta_2=0$ $\delta_2=1$
$H^4_{8,2}$	xt^{-n}	$t^{1/m}J_2$	$x(J_3-\beta m^{-1}\ln t)$ $x(J_3-{}^1/_2\beta m^{-1}n^{-1}(\ln x)^2)$	$m\neq 0$	$\delta_2=0$ $\delta_2=1$
$H^4_{8,3}$	x	$J_2-\alpha^{-1}\ln t$	$J_3+\beta\alpha^{-1}x\ln x\ln t$ $x(J_3-2\beta t)$	$m=-\alpha\neq 0$ —	$\delta_2=1$ $\delta_2=0$
$H^4_{9,1}$	$xe^{-\alpha t}$	J_2+2t	$x(J_3-\beta\alpha^{-1}(\ln x)^2$	$\alpha\neq 0$	$\delta_2=1$
$H^4_{9,2}$	x	J_2+2t	$J_3-2\beta x\ln t$	$\alpha=0$	$\delta_2=1$
H^4_{10}	$te^{2\alpha x}$	J_2+2x	$J_3-\beta x^2$	—	$\delta_1=0$
H^4_{11}	$x-kt$	J_2+2t	$J_3-\beta k^{-1}x^2$	$\alpha=0$	$\delta_1=0$
H^4_{12}	$te^{2\alpha x}$	J_2+2x	J_3+kx	$\beta=0$	$\delta_1=0$
$H^4_{13,1}$	t	$J_2+2\alpha^{-1}\ln x$	$x(J_3+k\alpha^{-1}\ln x)$	$\beta=0$ $\alpha\neq 0$	$\delta_2=1$
$H^4_{13,2}$	xt^{-n}	$J_2+m^{-1}\ln t$	$x(J_3+{}^1/_2km^{-1}\ln t)$	$\beta=0$	$\delta_2=1$
H^4_{14}	x	$J_2-\alpha^{-1}\ln t$	$J_3-{}^1/_2\alpha^{-1}\ln t$	$\alpha\neq 0$ $\beta=0$	$\delta_2=0$
H^4_{15}	$xe^{-\alpha t}$	J_2+2t	$x(J_3+kt)$	$\beta=0$	$\delta_2=1$
H^4_{16}	$x-mt$	J_2+2t	J_3+kt	$\alpha=\beta=0$	$\delta_1=0$

not zero simultaneously. Then the compatibility conditions for (2.1) give

$$\xi^\circ = 2ct + c_1, \quad \xi^1 = (c + c_0)\, x + c_2,$$
$$\eta^\circ = (bT + b_1)\, 2c_0,$$
$$\eta^1 = (c + c_0 + c_4)\, v - c_5 x + c_3 \quad (\delta_2 = 0),$$
$$\eta^1 = (c + c_0 + c_4)\, v - c_5\, x \ln x + c_3 x \quad (\delta_2 = 1),$$
$$\eta^2 = (2bc_0 + c_0 - c)\, \theta, \quad \eta^3 = c_4\varepsilon + c_5, \quad \delta_1 c_2 = 0,$$
$$\delta_1 B = 0,$$
$$\frac{1}{f}\frac{df}{dT} = \frac{1}{bT + b_1}, \quad (2.6)$$
$$\left(\varepsilon\frac{\partial\Phi}{\partial\varepsilon} - \Phi\right) c_4 + \frac{\partial\Phi}{\partial\varepsilon} c_5 = \left[(bT + b_1)\frac{\partial\Phi}{\partial T} - \Phi\right] 2c_0 = B$$

(b, b_1, and B are constants). The latter two equations define the forms of $\varphi(\varepsilon, T)$ and $f(T)$ with accuracy to the transformations of (2.4):

$$\Phi_1 = T^{\alpha+\beta}\Psi(\varepsilon T^{-\beta}), \qquad f = T^\alpha.$$

Here $\alpha c_4 = 2\beta c_0$, $c_5 = 0$, $b = 1/\alpha$. System S_1 allows the operators of (2.3) and also the linearly independent operator

$$X_5 = \alpha x\frac{\partial}{\partial x} + 2T\frac{\partial}{\partial T} + (\alpha + 2\beta)\, v\frac{\partial}{\partial v}.$$

Here α and β are arbitrary constants, $\alpha \neq 0$. It will be shown that $\alpha = 0$ is permissible in this case and in the following forms of the functions.

An additional extension transformation that preserves system S_1 has the form

$$t' = t, \quad x' = a_5{}^\alpha x, \quad T' = a_5{}^2 T, \quad v' = a_5^{(\alpha+2\beta)} v.$$

In the case

$$\Phi_2 = e^{(\alpha+\beta)}\Psi(\varepsilon e^{-\beta T}), \qquad f = e^{\alpha T}$$

we have $\alpha c_4 = 2\beta c_0$, $c_5 = 0$, $b = 0$, $b_1 = 1/\alpha$. The additional operator allowed by S_2 is

$$X_5 = \alpha x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial T} + (\alpha + 2\beta)\, v\frac{\partial}{\partial v}.$$

Here, for $\alpha \neq 0$, we may, from (2.4), assume that $\alpha = 1$. The additional transformation that preserves S_2 is

$$t' = t, \quad x' = e^{\alpha a_5} x, \quad T' = T + 2a_5, \quad v' = e^{(\alpha+2\beta)a_5} v.$$

In the case

$$\Phi_3 = T^\alpha\Psi(\varepsilon - \beta\ln T), \qquad f = T^\alpha$$

we have $\alpha c_5 = 2\beta c_0$, $c_4 = 0$, $b = 1/\alpha$, $b_1 = 0$. System S_3 allows, in addition to (2.3), the operator

$$X_5 = \alpha x\frac{\partial}{\partial x} + 2T\frac{\partial}{\partial T} + (\alpha v - 2\beta x)\frac{\partial}{\partial v} \quad (\delta_2 = 0),$$
$$X_5 = \alpha x\frac{\partial}{\partial x} + 2T\frac{\partial}{\partial T} + (\alpha v - 2\beta x\ln x)\frac{\partial}{\partial v} \quad (\delta_2 = 1).$$

The corresponding finite transformations are

$$t' = t, \quad x' = e^{\alpha a_5} x, \quad T' = e^{2a_5} T,$$
$$v' = e^{\alpha a_5}(v - 2\beta a_5 x) \quad (\delta_2 = 0),$$
$$t' = t, \quad x' = e^{\alpha a_5} x, \quad T' = e^{2a_5} T,$$
$$v' = e^{\alpha a_5}(v - \beta a_5 x(\alpha a_5 + 2\ln x)) \quad (\delta_2 = 1).$$

In the case

$$\Phi_4 = e^{\alpha T}\Psi(\varepsilon - \beta T), \qquad f = e^{\alpha T}$$

we have

$$\alpha c_5 = 2\beta c_0, \quad c_4 = 0, \quad b = 0, \quad b_1 = 1/\alpha.$$

The additional operator is

$$X_5 = \alpha x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial T} + (\alpha v - 2\beta x)\frac{\partial}{\partial v} \quad (\delta_2 = 0),$$
$$X_5 = \alpha x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial T} + (\alpha v - 2\beta x\ln x)\frac{\partial}{\partial v} \quad (\delta_2 = 1).$$

Here, of course, for $\alpha \neq 0$ we may put $\alpha = 1$. The corresponding transformations are

$$t' = t, \quad x' = e^{\alpha a_5} x, \quad T' = T + 2a_5,$$
$$v' = e^{\alpha a_5}(v - 2\beta a_5 x) \quad (\delta_2 = 0),$$
$$t' = t, \quad x' = e^{\alpha a_5} x, \quad T' = T + 2a_5,$$
$$v' = e^{\alpha a_5}[v - \beta a_5 x(\alpha a_5 + 2\ln x)] \quad (\delta_2 = 1).$$

For planar flows we have two further cases:

$$\Phi_5 = K\ln T + \Psi(\varepsilon T^\alpha), \quad f = T^\alpha \quad (K = \mathrm{const}).$$

In this case $c = -2c_0$, $c_5 = 0$, $b = 1$, $b_1 = 1/\alpha$. The additional operator X_5 and the finite transformations are obtained from the corresponding expressions for Φ_1 with $\beta = -\alpha$:

$$\Phi_6 = KT + \Psi(\varepsilon e^{\alpha T}), \quad f = e^{\alpha T} \quad (K = \mathrm{const}).$$

Here $c_4 = -2c_0$, $c_5 = 0$, $b = 0$, $b_1 = 1/\alpha$. The additional operator X_5 and the finite transformations are obtained from the corresponding expressions for Φ_1 with $\beta = -\alpha$.

If $\Phi(\varepsilon, T)$ is arbitrary, we have for the $f(T)$ satisfying the first condition of (2.5) that $c_0 = c_4 = c_5 = 0$, i.e., the same expressions as for arbitrary $f(T)$.

Consider now the possibility of extending the group for an $f(T)$ satisfying the second condition of (2.5). System (2.1) gives the following functions (with φ° and φ unknown functions of t and x) for f = constant:

$$\xi^\circ = 2ct + c_1, \quad \xi' = cx + c_2, \quad \eta^\circ = c_0 T + \varphi^\circ(t, x),$$
$$\eta^1 = (c + c_4)\, v + \varphi(t, x), \quad \eta^2 = (c_0 - c)\,\theta + f\partial\varphi^\circ/\partial x,$$
$$\eta^3 = c_4\varepsilon - \frac{\partial\varphi}{\partial x} + \frac{\delta_2}{x}\varphi, \quad \delta_1 c_2 = 0,$$
$$\frac{\partial\varphi^\circ}{\partial t} - f\frac{\partial^2\varphi^\circ}{\partial x^2} - \frac{\delta_1}{x} f\frac{\partial\varphi^\circ}{\partial x} = 0,$$
$$F_2\left(\frac{\partial^2\varphi}{\partial x^2} - \frac{\delta_2}{x}\frac{\partial\varphi}{\partial x} + \frac{\delta_2}{x^2}\varphi\right) + F_1\frac{\partial\varphi^\circ}{\partial x} - \frac{\partial\varphi}{\partial t} -$$
$$- \frac{\delta_1 + \delta_2}{x}\left[F_2\left(c_4\varepsilon - \frac{\partial\varphi}{\partial x} + \frac{\delta_2}{x}\varphi\right) -\right.$$
$$\left. - F_1(c_0 T + \varphi^\circ) - \Phi c_4\right] = 0, \quad (2.7)$$

$$(c_0T + \varphi_0)\frac{\partial F_2}{\partial T} + \left(c_4\varepsilon - \frac{\partial\varphi}{\partial x} + \frac{\delta_2}{x}\varphi\right)\frac{\partial F_2}{\partial \varepsilon} = 0, \quad (2.7)$$

(cont'd)

$$(c_0T + \varphi^\circ)\frac{\partial F_1}{\partial T} + \left(c_4\varepsilon - \frac{\partial\varphi}{\partial x} + \frac{\delta_2}{x}\varphi\right)\frac{\partial F_1}{\partial \varepsilon} = F_1(c_4 - c_0).$$

Then it follows that, if $\Phi(\varepsilon, T)$ is not a solution of

$$\frac{\partial^2\Phi}{\partial\varepsilon^2}\frac{\partial^2\Phi}{\partial T^2} - \left(\frac{\partial^2\Phi}{\partial\varepsilon\,\partial T}\right)^2 = 0, \quad (2.8)$$

we have

$$\varphi^\circ = c_6,\ \varphi = -c_5x + c_3\ (\delta_2 = 0),$$

$$\varphi = -c_5\,x\ln x + c_3x \quad (\delta_2 = 1).$$

For Φ we get from (2.7) the equation

$$\left(\varepsilon\frac{\partial\Phi}{\partial\varepsilon} - \Phi\right)c_4 + \frac{\partial\Phi}{\partial\varepsilon}c_5 + T\frac{\partial\Phi}{\partial T}c_0 + \frac{\partial\Phi}{\partial T}c_6 = B,$$

$$\delta_1 B = 0. \quad (2.9)$$

We have $c_0 = c_4 = c_5 = c_6 = 0$ if Φ is an arbitrary function, and then there is no scope for extending the group via f.

Comparison of (2.9) with the latter equations of (2.6) shows that (2.9) defines the forms of Φ_1, Φ_2, and Φ_3 above, in which, however (and in the corresponding operators and transformations) we have $\alpha = 0$, and in Φ_2 we may assume that $\beta = 0.1$ by virtue of (2.4). Further, (2.9) in the planar case gives a further value of Φ:

$$\Phi_7 = K\ln T + \Psi(\varepsilon - \beta\ln T), \qquad f = \text{const}.$$

Here $c_5 = \beta c_0$, $c_4 = 0$, $c_6 = 0$. The additional operator X_5 and the transformation are obtained from the expressions for Φ_3 with $\alpha = 0$.

Now let $\Phi(\varepsilon, T)$ satisfy (2.8). From the solutions to this equation we select those that satisfy (2.7). It can be shown that the solutions that do not satisfy

$$\frac{\partial^2\Phi}{\partial\varepsilon\,\partial T} + \beta\frac{\partial^2\Phi}{\partial\varepsilon^2} = 0 \quad (2.10)$$

lead again to (2.9) and to the corresponding φ° and φ. Other values of φ° and φ may be obtained for the Φ that satisfies (2.8) and (2.10):

$$\Phi_8 = KT + \Psi(\varepsilon - \beta T), \qquad f = \text{const}.$$

Then (2.7) gives

$$K\frac{\partial\varphi^\circ}{\partial x} + \frac{\partial\varphi}{\partial t} + \frac{\delta_1 + \delta_2}{x}K\varphi^\circ = 0, \qquad \frac{\partial\varphi}{\partial x} - \frac{\delta_2}{x}\varphi + \beta\varphi^\circ = 0,$$

$$\frac{\partial\varphi^\circ}{\partial t} - f\frac{\partial^2\varphi^\circ}{\partial x^2} - \frac{\delta_1}{x}f\frac{\partial\varphi^\circ}{\partial x} = 0, \quad (2.11)$$

and the conditions $c_4 = 0$, $c_0 = 0$ for the arbitrary constants, except for the values $K = \beta = 0$, for which c_0 remains arbitrary. In the latter case S allows an infinite group, which is natural, since it splits up into two independent equations, of which one is linear. The Lie algebra is generated by the operators of (2.3) and

$$X_5 = T\frac{\partial}{\partial T}, \qquad X^\circ = \varphi^\circ(t, x)\frac{\partial}{\partial T}$$

in which φ° is the solution to

$$\frac{\partial\varphi^\circ}{\partial t} - f\frac{\partial^2\varphi^\circ}{\partial x^2} - \frac{\delta_1}{x}f\frac{\partial\varphi^\circ}{\partial x} = 0. \quad (2.12)$$

Consider now (2.11) for the general case. We can obtain solutions $\varphi^\circ(tx)$ and $\varphi(tx)$ giving values of the coordinates different from the coordinates of the operator of group G either for planar flows or for $K = 0$. Consider first planar flows. If β, f, and K are independent, we get from (2.11) that

$$\varphi^\circ = c_7x + c_6, \qquad \varphi = -\beta\left(\tfrac{1}{2}c_7x^2 + c_6x\right) - Kc_7t + c_3.$$

The Lie algebra of the basic group is generated by six linearly independent operators, namely those of (2.3) and

$$X_5 = \frac{\partial}{\partial T} - \beta x\frac{\partial}{\partial v}, \qquad X_6 = x\frac{\partial}{\partial T} - \left(\beta\frac{x^2}{2} + Kt\right)\frac{\partial}{\partial v}.$$

From (2.4) we may assume $K = 1$ or $\beta = 1$. The corresponding transformations are

$$t' = t, \quad x' = x, \quad T' = T + a_5, \quad v' = v - \beta a_5x,$$

$$t' = t, \quad x' = x, \quad T' = T + a_6x,$$

$$v' = v - a_6\left(\tfrac{1}{2}\beta x^2 + Kt\right).$$

If $K = f\beta$, the system allows an infinite group; to the operators of (2.3), we add a manifold of the following form, in which φ° is the solution to (2.12) for $\delta_1 = 0$:

$$X^\circ = \varphi^\circ(t, x)\frac{\partial}{\partial T} - \beta\int\varphi^\circ(t, x)\,dx\,\frac{\partial}{\partial v}.$$

Now consider the case $K = 0$; Φ_8 coincides with Φ_4 for $\alpha = 0$, but the value $\alpha = 0$ may be shown to be special. The solution for the planar case is correct for any K, so it is sufficient to consider the other two types of flow. The general solution to (2.11) is of the form

$$\varphi^\circ = c_7\ln x + c_6,$$

$$\varphi = -\beta c_7(x\ln x - x) - \beta c_6x + c_3 \qquad (\delta_1 = 1,\ \delta_2 = 0),$$

$$\varphi = x\left[c_3 - \tfrac{1}{2}c_7\beta(\ln x)^2 - c_6\beta\ln x\right] \qquad (\delta_1 = \delta_2 = 1).$$

The basic group has six parameters. System S_4 for $\alpha = 0$ allows the following operator in addition to X_1, ..., X_5:

$$X_6 = \ln x\frac{\partial}{\partial T} - \beta(x\ln x - x)\frac{\partial}{\partial v} \qquad (\delta_2 = 0),$$

$$X_6 = \ln x\frac{\partial}{\partial T} - \frac{1}{2}\beta x(\ln x)^2\frac{\partial}{\partial v} \qquad (\delta_2 = 1).$$

The corresponding finite transformations are

$$t' = t, \quad x' = x \quad T' = T + a_6\ln x,$$

$$v' = v + a_6\beta(x\ln x - x) \quad (\delta_2 = 0),$$

$$t' = t, \quad x' = x, \quad T' = T + a_6\ln x,$$

$$v' = v - \tfrac{1}{2}a_6\beta x(\ln x)^2 \quad (\delta_2 = 1).$$

§3. We use this basic group of transformations to find particular solutions for S and S_1. The invariant solutions of unit rank are possible only in one-parameter subgroups. To find all invariant solutions it is sufficient to find the solutions essentially different relative to G_i.

We use the internal automorphism of G^i to construct an optimal system of one-parameter subgroups that allows us to find all the essentially different solutions for the above specializations S_i of system S. Comparison of the X_5 for S_1 and S_2, and also for S_3 and S_4, shows that the specific form of the coordinates allows us to show that the matrices of internal automorphisms of the basic groups for S_1 and S_2 (and S_3 and S_4) are identical, which facilitates construction of the optimal system.

We omit intermediate steps and give only the final form of the optimal system of one-parameter subgroups of the Si (i = 1, ..., 4) for nonspecial values of the parameters.

System S

$$H_1 = X_1, \quad H_2 = X_2, \quad H_3 = X_4,$$
$$H_4 = X_1 + X_3, \; H_5 = kX_2 + X_3,$$
$$H_6 = X_1 + X_2 + mX_3, \quad H_7 = X_4 + kX_3,$$
$$\delta_1 H_2 = \delta_1 H_5 = \delta_1 H_6 = 0, \quad (\delta_2 - 1) H_7 = 0.$$

System S_i (i = 1, ..., 4)

$$H_l^i = H_l \; (l = 1, \ldots, 7),$$

where, if $\beta \neq 0$, the general rule leads us to put in the operators

$$H_5^{1,2} \text{ and } H_7^{1,2} \; k = 1;$$

and in the operators

$$H_6^{1,2} \quad m = 0,1, \qquad H_6^{3,4}, \quad m = 0,$$
$$H_5^{3,4} = 0, \quad H_7^{3,4} = 0.$$

Further

$$H_8^i = mX_4 + X_5, \qquad H_9^i = X_1 + X_5,$$
$$H_{10}^i = X_2 - \alpha X_4 + X_5,$$
$$H_{11}^i = X_1 + kX_2 + X_5 \; (\alpha = 0),$$
$$H_{12}^i = X_2 + kX_3 - \alpha X_4 + X_5 \qquad (\beta = 0),$$
$$H_{13}^i = kX_3 + mX_4 + X_5 \qquad (\beta = 0),$$
$$H_{14}^{1,2} = X_3 - (\alpha + 2\beta) X_4 + X_5,$$
$$H_{14}^{3,4} = X_3 - \alpha X_4 + X_5 \; (\beta = 0),$$
$$H_{15}^i = X_1 + kX_3 + X_5 \; (\beta = 0),$$
$$H_{15}^{1,2} = X_1 + kX_3 + X_5 \; (\alpha = -2\beta, \; \delta_2 = 0),$$
$$H_{16}^i = X_1 + k_1X_2 + k_2X_3 + X_5 \qquad (\alpha = \beta = 0),$$
$$\delta_1 H_{10}^i = \delta_1 H_{11}^i = \delta_1 H_{12}^i = \delta_1 H_{16}^i = 0,$$
$$(1 - \delta_2) H_{13}^i = (1 - \delta_2) H_{15}^i = 0,$$
$$\delta_2 H_{14}^{1,2} = 0.$$

Here the H_j^i are the simplest representatives of the classes of operators of the one-parameter subgroups forming the optimal system for system S; k and m are arbitrary constants, $k \neq 0$. Operators that do not lead to invariant solutions are excluded. For convenience, operators that are not similar for any values of the physical parameters have been distinguished; the conditions under which the operator is not similar to the others are given in parentheses.

The invariant solutions found in the subgroups of H_j^i are given in the table; here $J_2(\xi)$ and $J_3(\xi)$ satisfy the system (S/H) of ordinary differential equations.

The table gives solutions for S_1, ..., S_4. It is readily seen that the solutions for S_5, ..., S_8 are to be found from this table also.

§4. Consider some of these invariant solutions.

1°. The steady-state flow and heat transfer in the annulus between coaxial cylinders of radius x_0 and x_1, and in the space between parallel planes. Solution H_1 is found by quadrature for arbitrary $\Phi(\varepsilon, T)$ and $f(T)$. System S/H has the form

$$\frac{\partial\Phi}{\partial\varepsilon} J_3'' - \frac{\delta_2}{\xi}\frac{\partial\Phi}{\partial\varepsilon} J_3' + \frac{\delta_2}{\xi^2}\frac{\partial\Phi}{\partial\varepsilon} J_3 - \frac{\partial\Phi}{\partial T} J_2' - \frac{\delta_1 + \delta_2}{\xi}\Phi = 0,$$
$$J_3' - \frac{\delta_2}{\xi} J_3 + \varepsilon = 0, \qquad fJ_2'' + f'J_2'^2 + \frac{\delta_1}{\xi} fJ_2' = 0. \quad (4.1)$$

A prime here denotes the derivative of a function with respect to its argument. We introduce the function $\chi(\tau_1, T)$ as follows:

$$\Phi = \tau_0(T) + \tau_1(\varepsilon, T), \quad \tau_1(0, T) = 0, \quad \varepsilon = \chi(\tau_1, T), \quad (4.2)$$

and all solutions will be examined in the case of interest to us:

$$\frac{d\tau_0}{dT} \leqslant 0, \qquad \frac{\partial\tau_1}{\partial\varepsilon} > 0, \qquad \frac{\partial^2\tau_1}{\partial\varepsilon^2} \leqslant 0, \qquad \frac{\partial\tau_1}{\partial T} \leqslant 0. \quad (4.3)$$

The temperature distribution is given by the well-known expressions

$$x = x_0 + h\frac{\vartheta(T, T_0)}{\vartheta(T_1, T_0)} \quad (\delta_1 = 0),$$
$$x = x_0 \exp\left(h\frac{\vartheta(T, T_0)}{\vartheta(T_1, T_0)}\right) \quad (\delta_1 = 1),$$
$$\vartheta(T, T_0) = \int_{T_0}^{T} f(T)\,dT, \qquad h = \begin{cases} x_1 - x_0 & (\delta_1 = 0), \\ \ln(x_1/x_0) & (\delta_1 = 1). \end{cases} \quad (4.4)$$

In this solution we may put T° = T, $f(T°) = \lambda(T)$.

Here T_0 and T_1 are the temperatures at the boundary surfaces x_0 and x_1, which either are given or are found from the conditions of heat transfer at these boundaries. Let the transfer at boundary x_1 be with a medium of constant temperature T_C and in accordance with $Q = R(T - T_C)$, in which R(z) is a function restricted by the conditions $R(-z) = -R(z)$, $dR/dz > 0$. Then it is easily shown that the equation for T_1 is

$$\vartheta(T_1, T_0) + x_1^{\delta_1} hR(T_1 - T_c) = 0$$

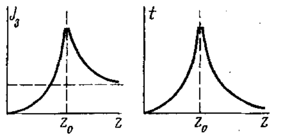
and that it has a unique solution. The same is true for the heat transfer at the other surface. The distributions of the stresses and velocities are

$$\Phi = \frac{P}{x_0^{\delta_1+\delta_2}} \exp\left[-(\delta_1 + \delta_2)\ln\frac{\vartheta(T, T_0)}{\vartheta(T_1, T_0)}\right],$$
$$v = x^{\delta_2}\left\{V - \frac{x_0^{\delta_1-\delta_2}}{\vartheta(T_1, T_0)} h \int_{T_0}^{T} \chi(T, P) f(T) \exp\left[h(\delta_1 - \delta_2)\frac{\vartheta(T, T_0)}{\vartheta(T_1, T_0)}\right] dT\right\}. \quad (4.5)$$

If x_1 is fixed, then V (the velocity at x_0, or the angular velocity if $\delta_2 = 1$) and P (the quantity expressed by the force or moment) are related by

$$V\vartheta(T_1, T_0) - x_0^{\delta_1-\delta_2} h \int_{T_0}^{T_1} \chi(T, P) f(T) \exp\left[h(\delta_1 - \delta_2)\frac{\vartheta(T, T_0)}{\vartheta(T_1, T_0)}\right] dT = 0\,, \qquad (4.6)$$

which has a unique solution because of condition (4.3) for P. For $\tau_0(T) \neq 0$, the flow may occupy the entire region $x_1 - x_0$ or only part of it, P being the decisive factor. In the latter case the integrations



in (4.5) and (4.6) are carried over the flow region ($\chi \equiv 0$ outside the flow), whose disposition is governed by the properties of the function

$$\tau_*(T) = \tau_0(T) x_0^{\delta_1+\delta_2} \exp\left[(\delta_1 + \delta_2) h \frac{\vartheta(T, T_0)}{\vartheta(T_1, T_0)}\right],$$

while the boundaries are given by $P - \tau_*(T) = 0$. From $\tau_*(T)$ we find that the flow region is adjacent to boundary x_0 for $T_1 < T_0$ subject to the condition $\tau_0'(T) < 0$. For $T_1 > T_0$ it is adjacent to the x_1 boundary, while in the axially symmetric case it may lie at either boundary or within the annulus.

2°. Flow in an unbounded medium $x_0 < x < \infty$ due to motion of a cylinder of finite radius $x_0 > 0$. Flow in a half plane. Solution $H^1_{3,1}$.

System (S/H) for J_2 and J_3 takes the form

$$J_2' - \frac{2}{\alpha^2}[2 + (1 + \delta_1)\alpha] J_2^{\alpha+1} = 0,$$

$$J_2^\beta z - \frac{1}{\alpha}[\alpha(\delta_2 - 1) - 2\beta] J_3 = 0,$$

$$J_3' + \frac{1}{\alpha}[(2 + \delta_1 + \delta_2)\alpha + 2\beta] J_2^{\alpha+\beta}\Psi(z) = 0\,.$$

The general solution is found in finite form for arbitrary $\Psi(z)$:

$$J_3 = z\exp(-\beta\varkappa_1(\alpha)Z(z)),$$

$$\xi = \alpha\varkappa_1^{-1}(\alpha)\,[c_0 - (\delta_2 - 1 - 2\beta/\alpha)^{-\alpha/\beta}\exp(\alpha x_1(\alpha)Z(z))$$

$$\text{for}\quad \varkappa_1 \neq 0,$$

$$J_3 = c_0^{-\beta/\alpha}(\delta_2 - 1 - 2\beta/\alpha)^{-1}z,$$

$$\xi = -c_0\alpha^2 Z(z) \quad \text{for} \quad \varkappa_1 = 0\,,$$

$$J_2 = \left(c_0 - \frac{\varkappa_1(\alpha)}{\alpha}\xi\right)^{-1/\alpha},$$

$$Z(z) = \int\frac{dz}{\beta\varkappa_1(\alpha)z - \varkappa_2(\alpha,\beta)\Psi(z)}\,,$$

$$\varkappa_1(\alpha) = 2\,[2 + (1 + \delta_1)\alpha],$$

$$\varkappa_2(\alpha, \beta) = [2\beta + \alpha(1 - \delta_2)]\,[(2 + \delta_1 + \delta_2)\alpha + 2\beta]. \qquad (4.7)$$

Here $\delta_2 - 1 - 2\beta/\alpha > 0$, since $\delta_2 - 1 - 2\beta/\alpha = 0$ describes the motion of the medium as a solid body. Consider the solution that satisfies the conditions

$$T(x, 0) = 0, \qquad \lim_{x\to\infty} T(x, t) = 0, \qquad \lim_{x\to\infty} v(x, t) = 0,$$

$$\lim_{x\to\infty}\left(x^{\delta_1} f(T)\frac{\partial T}{\partial x}\right) = 0, \qquad \lim_{x\to\infty}(x^{\delta_1+\delta_2}T^{\alpha+\beta}\Psi(z)) = 0\,, \qquad (4.8)$$

of which the last two represent the fact that the total heat flux and total force or moment are zero at infinity. These conditions are, from (4.3), complied with for the values

$$-\frac{2}{1+\delta_1} < \alpha < 0, \qquad \beta > -\frac{\alpha(2 + \delta_1 + \delta_2)}{2},$$

$$\Psi(0) = 0. \qquad (4.9)$$

Conditions (4.8) may impose some restrictions also on $\Psi(z)$.

The solution satisfying these conditions has the form

$$T = \left(-\frac{\varkappa_1(\alpha)}{\alpha}\right)^{-1/\alpha} x^{2/\alpha} t^{-1/\alpha}, \qquad v = x^{(\alpha+2\beta)/\alpha} z\exp(-\beta\varkappa_1(\alpha)Z(z)),$$

$$t = -\alpha\varkappa_1^{-1}(\alpha)(\delta_2 - 1 - 2\beta/\alpha)^{-\alpha/\beta}\exp(\alpha\varkappa_1(\alpha)Z(z)).$$

This describes the flow in a medium of zero initial temperature heated at the boundary x_0 by a heat flux $Q = -2\alpha^{-1}x_0^{-1}T^{\alpha+1}(x_0, t)$ the body moving with a velocity $V_0 = AJ_3(t)$.

This solution shows that $\varkappa_1 > 0$, $\varkappa_2 > 0$ as a consequence of (4.3) and (4.9); the behavior of the solution at small z is dependent on that of $\Psi(z)$.

For $\varkappa_2(\alpha, \beta)\Psi'(0) > \beta\varkappa_1(\alpha)$ we get both branches of the curves; Fig. 1 shows the case $\varkappa_2\Psi'(0) < \varkappa_1(\beta - \alpha)$; while the case $\varkappa_2\Psi'(0) \geqslant \varkappa_1(\beta - \alpha)$ differs only in the form of t(z) for z small.

For $\varkappa_2(\alpha, \beta)\Psi'(0) < \beta\varkappa_1(\alpha)$ we get only the right branch, and here we should put $z_0 = 0$ (z_0 is the root of $\beta\varkappa_1(\alpha)z - \varkappa_2(\alpha, \beta)\Psi(z) = 0$).

Motion from a state of rest corresponds to the left branch of the curve; the right branch corresponds to a flow when the medium is initially in motion.

The case $\alpha = -2/(1 + \delta_1)$, $\beta > (2 + \delta_1 + \delta_2)/(1 + \delta_1)$ describes the transient-state flow in a steady temperature distribution; this and the velocity distribution are described by

$$T = Cx^{-(1+\delta_1)}, \qquad v = \frac{C^\beta}{\delta_2 - 1 + \beta(1 + \delta_1)} x^{1-\beta(1+\delta_1)} z,$$

$$t = \frac{4C^{\frac{2}{1+\delta_1}}}{(1 + \delta_1)^2\varkappa_2}\int\frac{dz}{\Psi(z)}\,.$$

For $\alpha = -2/(1 + \delta_1)$, $\beta = (2 + \delta_1 + \delta_2)/(1 + \delta_1)$ there are steady-state velocity and temperature distributions.

3°. Axially symmetric flow in an undoubted medium due to sources of heat and torque located at the axis. Certain other flows.

Solution $H^1_{3,2}$. The system (S/H) takes the form

$$J_2^\alpha J_2'' + \alpha J_2^{\alpha-1}J_2'^2 + \delta_1\frac{1}{\xi}J_2^\alpha J_2' + \frac{m + \alpha}{2m}J_2'\xi - \frac{1}{m}J_2 = 0\,,$$

$$J_2^\alpha\Psi' J_3'' - J_2^\alpha\Psi' J_3'\left(\beta'\frac{J_2'}{J_2} + \frac{\delta_2}{\xi}\right) + \frac{m + \alpha}{2m}\xi J_3' +$$

$$+\left[J_2^\alpha\Psi'\left(\frac{\beta J_2'}{J_2} + \frac{1}{\xi}\right)\frac{\delta_2}{\xi} - \frac{m + \alpha + 2\beta}{2m}\right]J_3 - J_2^{\alpha+\beta}\Psi\left[(\alpha + \beta)\frac{J_2'}{J_2} + \frac{\delta_1 + \delta_2}{\xi}\right] = 0\,,$$

$$z = \left(-J_3' + \frac{\delta_2 J_3}{\xi}\right)J_2^{-\beta}. \qquad (4.10)$$

A full analysis will not be given for this; instead, we consider some particular solutions. See [7-9] for self-modeling solutions of this type for the energy equation for S_1 [the first equation of (4.10)]. Consider the following boundary-value problem for system S_1:

$$T(x, 0) = 0, \qquad \lim_{x\to 0}\left(-x^{\delta_1}T^\alpha\frac{\partial T}{\partial x}\right) = t^\gamma \quad (\gamma \geqslant 0),$$

$$\int_0^\infty T(x, t)\,x^{\delta_1}dx = \int_0^t\left\{\lim_{x\to 0} x^{\delta_1}\left(-T^\alpha\frac{\partial T}{\partial x}\right)\right\}dt \quad (-1 < \alpha < 0), \qquad (4.11)$$

$$v(x, 0) = 0, \qquad \lim_{x\to\infty} v(x, t) = 0, \qquad \lim_{x\to 0}[x^{\delta_1+\delta_2}\Phi(x, t)] = Pt^\nu,$$

$$\delta_2 = 1, \qquad \nu = \frac{\alpha + 1 + 2\alpha\gamma}{\alpha + 1}\,, \qquad \beta = 0, \qquad \gamma < -\frac{\alpha + 1}{2\alpha}\,. \qquad (4.12)$$

We introduce a $\chi(\psi)$ such that $\Psi = \psi_0 + \psi(z)$, $\psi(0) = 0$, $z = \chi(\psi)$; then system (4.10) may be put as

$$J_2^\alpha J_2'' + \alpha J_2^{\alpha-1} J_2'^2 + \frac{1}{\xi} J_2^\alpha J_2' + pJ_2'\xi - \frac{\gamma}{1+\alpha} J_2 = 0,$$

$$p = \frac{1+\alpha+\gamma\alpha}{2(\alpha+1)},$$

$$\psi' + \left(\alpha \frac{J_2'}{J_2} + \frac{2}{\xi}\right)(\psi + \psi_0) + pJ_2^{-\alpha}\xi\chi(\psi) = 0,$$

$$J_3' - \frac{1}{\xi} J_3 + J_2^\beta \chi(\psi) = 0. \qquad (4.13)$$

The boundary conditions for J_2 and J_3 are

$$\lim_{\xi\to 0}(-\xi J_2^\alpha J_2') = 1, \qquad \int_0^\infty J_2 \xi d\xi = \frac{1}{1+\gamma}, \qquad (4.14)$$

$$\lim_{\xi\to\infty}(\xi^2 J_2^\alpha \Psi) = P, \qquad \lim_{\xi\to\infty} J_3 = 0. \qquad (4.15)$$

Consider the integral curves for the first equation in (4.13). All the integral curves for (4.13) decrease monotonically and for $\psi_0 \neq 0$ meet the ξ axis at finite points. For $\psi_0 = 0$ we get the special point $\psi = 0$, $\xi = 0$ (saddle point), with the ξ axis as the integral curve. For $\xi \to \infty$ the functions $\psi(\xi)$ and $J_2(\xi)$ tend asymptotically to zero as

$$\psi = cJ_2^{-\alpha}(\xi)\xi^{-2}\exp\left[-p\chi'(0)\int J_2^{-\alpha}(\xi)\xi d\xi\right],$$

$$J_2(\xi) = a\xi^{2/\alpha} + 0(\xi^{2/\alpha}),$$

$$a = [-4(1+1/\alpha)]^{-1/\alpha}.$$

Hence the total moment at infinity is zero for $\psi_0 = 0$ and $\chi'(0) \neq 0$, whereas it is finite if only $\chi'(0) = 0$. For $\psi \to \infty$ the function has the form $\chi(\psi) = B\psi^N$ $(N > 1)$. The integral curves of (4.13) behave as follows near the ψ axis:

$$\psi = J_2^{-\alpha}\xi^{-2}\left[c + pB(N-1)\int_0^\xi J_2^{-\alpha N}\xi^{3-2N}d\xi\right]^{-\frac{1}{N-1}}.$$

From (4.15), to each P there corresponds one (and only one) integral curve for each P if $N < 2$; for $N \geq 2$ there are no integral curves that satisfy the first condition of (4.15). From $\psi(\xi)$ we find the J_3 that satisfies the second condition of (4.15):

$$J_3 = \xi\int_\xi^\infty \chi(\psi)\xi^{-1}d\xi.$$

The flow region is finite $(0 \leq \xi < \xi_*)$ for $\psi_0 \neq 0$, in which ξ_* is the point where the $\psi(\xi)$ curve meets the abscissa. The temperature and velocity are given by

$$T = t^{\frac{\gamma}{\alpha+1}} J_2(\xi), \quad v = t^p J_3(\xi), \quad \xi = xt^{-p} \quad \left(p = \frac{\alpha+1+\gamma\alpha}{2(1+\alpha)}\right).$$

For the case $\psi_0 = 0$ we have for large x and small t that

$$T \sim E_1 x^{\frac{2}{\alpha}} t^{-\frac{1}{\alpha}}, \qquad v \sim E_2 x^{-4r+1} t^{4rp} \quad \text{for} \quad \chi_{(0)}^{(r)} \neq 0, \quad r > 1,$$
$$v \sim E_3 x^{-3+2s} t^{2p(2-s)}, \quad s = (1+\gamma+1/\alpha)\chi'(0) \quad \text{for} \quad r = 1.$$

The solution to (4.10) subject to (4.11) for $\beta = 0$, $\gamma = -(\alpha+1)/\alpha$ may be treated as a steady-state flow in a bounded region $x_0 < x < x_1$ due to rotation or to translational motion of a cylinder, or as a flow between parallel planes provided that at boundary x_0 there is an influx of heat $Q_0 = qt^\gamma$, the thermal conductivity of the external medium being equal to that of the medium under study. If x_1 is fixed, while the velocity V at x_0 is given, then the temperature and velocity are

$$T = t^{-\frac{1}{\alpha}} J_2(x),$$

$$v = x^{\delta_2}\int_x^{x_1} \chi(\psi, P)x^{-\delta_2}dx, \quad \psi = PJ_2^{-\alpha}(x)x^{-(\delta_1+\delta_2)} - \psi_0$$

and P is defined by

$$V = x_0^{\delta_2}\int_{x_0}^{x_1} \chi(\psi, P)x^{-\delta_2}dx.$$

For $\psi_0 \neq 0$ a flow arises subject to the condition $P > \psi_0 J_2^\alpha(x_0)x_0^{\delta_1+\delta_2}$, while for $P > \psi_0 J_2^\alpha(x_1)x_1^{\delta_1+\delta_2}$ this flow fills the entire region $x_0 < x < x_1$; for $\psi_0 x_0^{\delta_1+\delta_2} J_2^\alpha(x_0) < P < \psi_0 x_1^{\delta_1+\delta_2} J_2^\alpha(x_1)$ the flow occurs in a region $x_0 < x < x_*$, in which x_* is the root of $P = \psi_0 x^{\delta_1+\delta_2} J_2^\alpha(x)$, which is unique by virtue of the condition $\alpha < 0$.

The other solutions for S and S_1 are presented without detailed analysis.

Solution H_4 is represented in finite form for arbitrary $f(T)$ and $\Phi(\varepsilon, T)$:

$$x = c_0\int f(T)\,dT, \qquad \Phi = c - c_0\int f(T)\,dT \quad \text{for} \quad \delta_1 = 0,$$

$$x = \exp\left(c_0\int f(T)\,dT\right), \qquad \Phi = c\exp\left[-c_0(\delta_1+\delta_2)\int f(T)\,dT\right] -$$
$$- \frac{1}{2\delta_2+1+\delta_1}\exp\left[c_0(\delta_2+1)\int f(T)\,dT \quad \text{for} \quad \delta_1 = 1,\right.$$

$$v = x^{\delta_2}\{t - c_0\int\chi(T, c)f(T)\exp[c_0(\delta_1-\delta_2)\int f(T)dT]\},$$

in which χ is given by (4.2).

Solution H_5 describes isothermal steady-state flow between two parallel planes:

$$T = \text{const}, \qquad v = k^{-1}x + c.$$

Solution H_6 represents uniformly propagating waves. After introducing the $\chi(\tau, T)$ of (4.2), the system reduces to one equation of first order:

$$\frac{d\tau_1}{dJ_2} = \frac{\chi(\tau_1, J_2) + m}{J_2 - c} f(J_2) - \frac{d\tau_0(J_2)}{dJ_2},$$

$$J_3 = -\int\frac{\chi(J_2)}{c - J_2} f(J_2)\,dJ_2, \quad \xi = \int\frac{f(J_2)\,dJ_2}{c - J_2}.$$

Solutions H_3 and H_7 are self-modeling. System (S/H) is of the form

$$f(J_2)J_2'' + f'(J_2)J_2'^2 + \frac{\delta_1}{\xi}f(J_2)J_2' + \frac{1}{2}J_2'\xi = 0,$$

$$\frac{\partial\Phi}{\partial\varepsilon}J_3'' - \frac{\delta_1\Phi}{\xi} - \frac{\partial\Phi}{\partial T}J_2' + \frac{1}{2}J_3'\xi - \frac{1}{2}J_3 = 0,$$

$$\varepsilon = -J_3' \quad (H_3),$$

$$\frac{\partial\Phi}{\partial\varepsilon}\xi J_3'' - \frac{2\Phi}{\xi} - \frac{\partial\Phi}{\partial T}J_2' + \frac{1}{2}\xi^2 J_3' + \frac{\partial\Phi}{\partial\varepsilon}J_3' = 0,$$

$$\varepsilon = -k - J_3'\xi \quad (H_7).$$

Solution H_2 is trivial: T = constant, v = constant.

Solution H_9^1. System (S/H) is of the form

$$J_2^\alpha J_2'' + \alpha J_2^{\alpha-1}J_2'^2 + \frac{\delta_1}{\xi}J_2^\alpha J_2' + \alpha J_2'\xi - 2J_2 = 0,$$

$$z = J_2^{-\beta}\left(-J_3' + \frac{\delta_2}{\xi}J_3\right),$$

$$\frac{d}{d\xi}[J_2^{\alpha+\beta}\Psi(z)] + \frac{\delta_1+\delta_2}{\xi}J_2^{\alpha+\beta}\Psi(z) + (\alpha+2\beta)J_3 - \alpha\xi\frac{dJ_3}{d\xi} = 0.$$

The substitution $J_2 = \xi^{2/\alpha} J(\zeta)$, $\zeta = \ln \xi$ and the introduction of the new function $Z(J) = dJ/d\zeta$ convert the first equation (for $\alpha \neq 0$) to a first-order equation (this is a standard equation for $\alpha = 0$).

Solution H^1_{10}. System (S/H) is of the form

$$4\alpha^2\xi^2 J_2^{\alpha} J_2'' + 4\alpha^3\xi^2 J_2^{\alpha-1} J_2'^2 + [4(3\alpha+2)\alpha\xi J_2^{\alpha} - 1] J_2' + 4(\alpha+1) J_2^{\alpha+1} = 0,$$

$$2\alpha\xi \frac{d}{d\xi}[J_2^{\alpha+\beta}\Psi(z)] + 2(\alpha+\beta) J_2^{\alpha+\beta}\Psi(z) + J_3' = 0,$$

$$z = -2(\beta J_3 + \alpha J_3'\xi) J_2^{-\beta}.$$

The substitution $J_2 = \xi^{-1/\alpha} J(\zeta)$, $\zeta = \ln \xi$ and introduction of the new function $Z(J) = dJ/d\zeta$ convert the first equation (for $\alpha \neq 0$) to a first-order equation. For $\alpha = 0$ the system integrates in quadratures for an arbitrary $\Psi(z)$. The temperature and velocity are

$$T = c_0 \exp[2(x+2t)], \qquad v = -\frac{1}{2\beta} z \exp\left(2\beta x - \int \frac{dz}{z - \beta\psi(z)}\right),$$

$$t = -\frac{1}{4\beta}\int \frac{dz}{z - \beta\Psi(z)} - \frac{1}{4}\ln c_0 .$$

Solution H^1_{11}. The equation for J_2 in system (S/H) is integrated in quadratures

$$J_2 = c_1 e^{\lambda_1\xi} + c_2 e^{\lambda_2\xi}, \qquad \lambda_{1,2} = -k/2f \pm \sqrt{(k/2f)^2 + 2/f} .$$

The second equation takes the form

$$\frac{d}{d\xi}[J_2^{\beta}\Psi(z)] - k\frac{dJ_3}{d\xi} + 2\beta J_3 = 0, \qquad z = -J_3' J_2^{-\beta} .$$

Solution $H^1_{13,1}$. The equation for J_2 in systems (S/H) coincides with the equation from $H^1_{8,2}$. The second equation takes the form

$$\frac{d}{d\xi}[J_2^{\alpha}\Psi(z)] + 2\frac{J_2^{\alpha}\Psi}{\xi} + \frac{k}{2m}\xi - \frac{m+\alpha}{2m} J_3'\xi^2 = 0, \qquad z = -J_3'\xi .$$

This reduces to a first-order equation after introduction of $\chi(\psi)$:

$$\psi' = -\left(\alpha\frac{J_2'}{J_2} + \frac{2}{\xi}\right)(\psi + \psi_0) - \frac{m+\alpha}{2m}\frac{\xi\chi(\psi)}{J_2^{\alpha}} - \frac{k}{2m}\frac{\xi}{J_2^{\alpha}} .$$

Solution $H^1_{13,2}$ is presented in quadratures for an arbitrary $\Psi(z)$. The temperature and velocity are

$$T = \left(\frac{\alpha x^2}{c_0\alpha - 4(\alpha+1)t}\right)^{1/\alpha},$$

$$v = x\left[c + \frac{k}{\alpha}\ln x + \frac{\alpha}{\alpha+1}\Psi\left(-\frac{k}{\alpha}\right)\ln\left(t - \frac{c_0\alpha}{4(\alpha+1)}\right)\right] \quad (\alpha \neq -1)$$

$$v = x\,[c - k\ln x - 4c_0^{-1}\Psi(k)t \quad (\alpha = -1) .$$

Solution $H^1_{14,1}$. The equation for J_2 in system (S/H) coincides with the equation from $H^1_{8,2}$. The second equation takes the form

$$\frac{d}{d\xi}[J_2^{\alpha+\beta}\Psi(z)] + \frac{\delta_1}{\xi} J_2^{\alpha+\beta}\Psi - \frac{1}{\alpha+2\beta}\left(\frac{1}{2} + \beta\xi J_3'\right) = 0 .$$

This reduces to a first-order equation after introduction of $\chi(\psi)$:

$$\psi' = -\left((\alpha+\beta)\frac{J_2'}{J_2} + \frac{\delta_1}{\xi}\right)(\psi + \psi_0) - \frac{\beta}{\alpha+2\beta}\frac{\chi(\psi)\xi}{J_2^{\alpha}} + \frac{1}{2(\alpha+2\beta)} J_2^{-(\alpha+\beta)}.$$

Solution $H^1_{14,2}$ is presented in quadratures for an arbitrary $\Psi(z)$. The temperature and velocity are

$$T = \left(\frac{\alpha x^2}{c_0 - \varkappa_1(\alpha)t}\right)^{1/\alpha},$$

$$v = \frac{1}{\alpha}\ln x - \frac{(1+\delta_1)\alpha^2}{\varkappa_1(\alpha)}\int \frac{\Psi(-(1/\alpha)J_2^{\alpha/2})}{J^{\alpha/2+1}} dJ_2,$$

$$J_2 = \left(\frac{\alpha}{c_0 - \varkappa_1(\alpha)t}\right)^{1/\alpha}.$$

Solution H^1_{15}. The equation for J_2 coincides with the equation from H^1_8. The second equation of system (S/H) takes the form

$$\frac{d}{d\xi}[J_2^{\alpha/2}\Psi(z)] + \frac{\delta_1}{\xi} J_2^{\alpha/2}\Psi(z) - \alpha J_3'\xi + k = 0 \quad (H^1_{15,1}),$$

$$\frac{d}{d\xi}[J_2^{\alpha}\psi(z)] + \frac{2}{\xi} J_2^{\alpha}\psi(z) - \alpha\xi^2 J_3' + k\xi = 0 \quad (H^1_{15,2}).$$

This reduces to a first-order equation after introduction of $\chi(\psi)$:

$$\psi' = -\left(\frac{\alpha}{2}\frac{J_2'}{J_2} + \frac{\delta_1}{\xi}\right)(\psi + \psi_0) - \frac{\alpha\xi}{J_2^{\alpha}}\chi(z) + kJ_2^{-\alpha/2} \quad (H^1_{15,1}),$$

$$\psi' = -\left(\alpha\frac{J_2'}{J_2} + \frac{2}{\xi}\right)(\psi + \psi_0) - \frac{\alpha\xi}{J_2^{\alpha}}\chi(\psi) - k\xi J_2^{-\alpha} \quad (H^1_{15,2}).$$

Solution H^1_{16} is presented in quadratures for an arbitrary $\Psi(z)$. The temperature and velocity are given as follows: $J_2(\xi)$ coincides with that found for H^1_{11}:

$$v = kt - \int \frac{\chi(\psi)d\psi}{m\chi(\psi)+k}, \qquad \xi = \int \frac{d\psi}{m\chi(\psi)+k} .$$

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